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LETTER TO THE EDITOR

Critical indices for an Ising model with pure triplet interactions on the triangular lattice

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Abstract. Extended series for the spontaneous magnetization and reduced susceptibility of the triplet model on the triangular lattice are analysed using Padé approximants. Baxter and Wu's exact solution for the specific heat has been used as a guide in interpreting the approximants and it is concluded that $\beta = 0{\cdot}080 \pm 0{\cdot}005$ and $\gamma' = 1{\cdot}15 \pm 0{\cdot}15$.

The Ising model with pure triplet interactions on the triangular lattice has been studied recently by Wood and Griffiths (1973) and Griffiths and Wood (1973) who derived series expansions and discovered a duality relation from which they conjectured the exact critical point. The subsequent exact solution in zero magnetic field by Baxter and Wu (1973) confirmed the critical point and showed that the specific heat diverges with exponent $\alpha' = \frac{2}{3}$. A detailed introduction to this model is given in the paper by Griffiths and Wood. This is a simple model in two dimensions which exhibits a phase transition and a study of its critical indices is of direct relevance to theories of scaling (Fisher 1967); especially since the results of Baxter and Wu establish that the critical behaviour is essentially different from the simple spin $\frac{1}{2}$ Ising model. In the absence of an exact solution for the spontaneous magnetization and susceptibility, series expansions may be used to estimate the corresponding critical indices. As with analogous studies of the simple Ising model, the existence of a known solution for one thermodynamic function (specific heat) provides a guide in assessing extrapolation procedures to be used on other functions.

As a by-product of recent extensive work on the simple Ising model (Sykes *et al* 1973b) extended series expansions for the triplet model have been obtained. The series are developments in powers of the low-temperature counting variable, $u = \exp(-4J_3/kT)$, where k is Boltzmann's constant, T is the absolute temperature and J_3 is the three-spin interaction energy (Griffiths and Wood use u_3 for u). The expansions for the reduced spontaneous magnetization $I(u)$ and reduced susceptibility $\chi_0(u)$ are available through u^{12}. The series to be studied are

$$I(u) = 1-2u^3-12u^4-66u^5-350u^6-1848u^7-9780u^8-52\,012u^9$$
$$-278\,118u^{10}-1495\,092u^{11}-8077\,130u^{12}-\ldots,$$
$$\chi_0(u) = u^3+12u^4+99u^5+726u^6+4968u^7+32\,664u^8+209\,238u^9$$
$$+1316\,610u^{10}+8178\,846u^{11}+50\,320\,232u^{12}+\ldots.$$

Apart from a misprint in the coefficient of u^8 in $\chi_0(u)$ the above series agree with Griffiths and Wood through u^9. The coefficients of u^{10} as quoted by Griffiths and Wood were slightly in error (Sykes *et al* 1973a) and have been corrected; the coefficients of u^{11} and u^{12} are new. The result of Baxter and Wu (1973) gives the zero-field partition function implicitly as the root of a quartic. This root has been expanded by Martin (unpublished) using a development of the Newton–Raphson iterative technique to obtain the specific heat series which we quote through order 20 as:

$$C(u) = 9u^3+48u^4+275u^5+1494u^6+8232u^7+45\,600u^8+254\,286u^9+1425\,050u^{10}$$
$$+8018\,670u^{11}+45\,269\,676u^{12}+256\,273\,290u^{13}+1454\,128\,704u^{14}$$
$$+8267\,220\,870u^{15}+47\,082\,196\,800u^{16}+268\,534\,387\,548u^{17}$$
$$+1533\,598\,103\,628u^{18}+8768\,539\,453\,614u^{19}$$
$$+50\,187\,192\,782\,740u^{20}+\ldots.$$

Following Griffiths and Wood (1973) we have used Padé approximants (Baker 1965) to estimate the critical indices. Specifically, knowing the critical point, $u_c = 3-2\sqrt{2}$, we have evaluated (u_c-u) Dlog approximants at $u = u_c$. The estimates obtained for the indices from diagonal, $[n/n]$, and paradiagonal, $[n\pm 1/n]$, approximants to $C(u)$, $I(u)$ and $\chi_0(u)$ are presented in tables 1–3. As is the usual practice (Gaunt and Guttmann 1974), those approximants which contain a 'spurious' pole within the physical circle are marked with a dagger. As the specific heat estimates are only tabulated for purposes of comparison with the other functions the series used was truncated after order 12. When an extended series (34 terms) is analysed we notice a linearity between the α' estimates and n^{-1}. This is shown in figure 1 in which the estimates obtained from $[n/n]$ approximants are plotted against n^{-1}. From table 1 we estimate $\alpha' = 0{\cdot}67\pm0{\cdot}03$. In this we have been helped by a knowledge of the estimates from the longer series in that we have used the last entry in table 1 as a form of upper limit. In table 2 we notice that

Table 1. Specific heat. Estimates for α' from (u_c-u) Dlog Padé approximants evaluated at $u = u_c = 3-2\sqrt{2}$.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
1		0·92612†	0·79284
2	0·81921	0·76834	0·74613
3	0·75323	0·72428	0·71867
4	0·71900	0·70724	

† Denotes presence of spurious pole inside physical circle.

Table 2. Spontaneous magnetization. Estimates for β from (u_c-u) Dlog Padé approximants evaluated at $u = u_c = 3-2\sqrt{2}$.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
1			−0·048299
2		−0·058302	−0·062609
3	−0·061477	−0·070900	−0·070159
4	−0·070151	−0·070606†	−0·074209
5	−0·073760	−0·074842	−0·074535
6	−0·074458		

† Denotes presence of spurious pole inside physical circle.

Table 3. Reduced susceptibility. Estimates for γ' from $(u_c - u)$ Dlog Padé approximants evaluated at $u = u_c = 3-2\sqrt{2}$.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
1		1·69608	1·63909
2	1·67935†	0·75310	1·39885
3	1·41998	1·37375	1·31078
4	1·34441	1·49563†	

† Denotes presence of spurious pole inside physical circle.

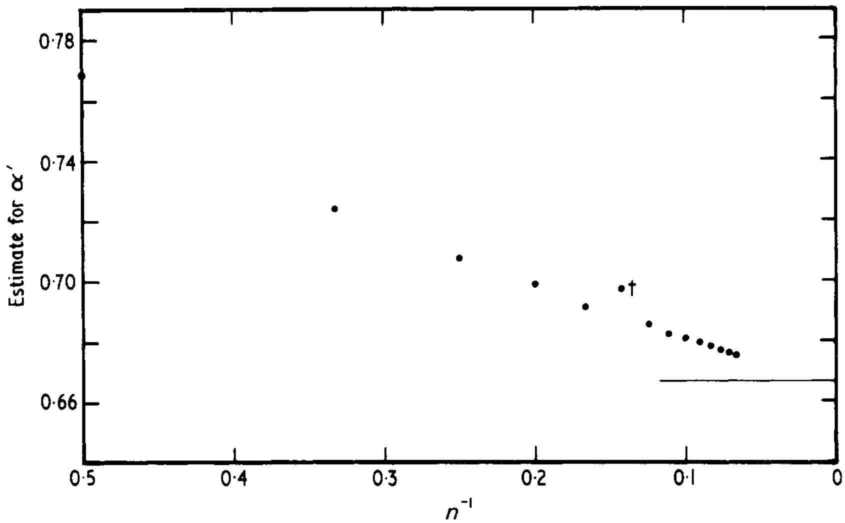

Figure 1. Specific heat. Estimates for α' from $[n/n]$ Padé approximants plotted against n^{-1}. The presence of a spurious pole inside the physical circle is denoted by †.

the estimates for β have a steady upward trend. Guided by the behaviour of the specific heat we estimate

$$\beta = 0{\cdot}080 \pm 0{\cdot}005. \tag{1}$$

Similarly, from the decreasing sequence in table 3, we obtain

$$\gamma' = 1{\cdot}15 \pm 0{\cdot}15. \tag{2}$$

In arriving at the above estimates we have assumed that a trend which develops early in the Padé table will persist. We have no theoretical justification for this assumption but rely solely on the fact that this holds for the specific heat. From figure 1 we see that the specific heat estimates converge smoothly towards the correct limit. However, the convergence is relatively slow and using the last entry in table 1 as an estimate for α' is clearly misleading. The uncertainty ranges quoted in equations (1) and (2) are large and reflect the belief that the convergence of the Padé estimates will also be slow for these exponents.

The estimates above are slightly different from those of Griffiths and Wood (1973) who quote $0{\cdot}070 \leqslant \beta \leqslant 0{\cdot}071$, $\frac{5}{4} \leqslant \gamma' \leqslant \frac{7}{5}$. Our estimates also differ in that, guided by the slow convergence of the estimates for the specific heat exponent, we quote wide uncertainty ranges. From the estimates in equations (1) and (2) we tentatively suggest the simple fractions $\beta = \frac{1}{12}$, $\gamma' = \frac{7}{6}$, which when combined with $\alpha' = \frac{2}{3}$ satisfy the scaling hypothesis (Fisher 1967)

$$\alpha' + 2\beta + \gamma' = 2.$$

Baxter (1974) has recently shown that the decay in correlation length is governed by an index $\nu = \frac{2}{3}$ so that the scaling relation

$$d\nu = 2 - \alpha$$

is exactly satisfied. If this model does obey the scaling laws we would predict the critical isotherm exponent to be $\delta = 15{\cdot}7 \pm 1{\cdot}0$ using equation (1) or $\delta = 15$ if we used $\beta = \frac{1}{12}$.

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References

Baker G A Jr 1965 *Adv. Theor. Phys.* **1** 1–58

Baxter R J 1974 *Ising Model on a Triangular Lattice with Three-Spin Interactions II. Free Energy and Correlation Length* preprint

Baxter R J and Wu F Y 1973 *Phys. Rev. Lett.* **31** 1294–7

Fisher M E 1967 *Rep. Prog. Phys.* **30** 615–730

Gaunt D S and Guttmann A J 1974 *Phase Transitions and Critical Phenomena* vol 3, ed C Domb and M S Green (New York: Academic Press)

Griffiths H P and Wood D W 1973 *J. Phys. C: Solid St. Phys.* **6** 2533–54

Sykes M F, Watts M G and Martin J L 1973a *J. Phys. C: Solid St. Phys.* **6** L461

Sykes M F *et al* 1973b *J. Math. Phys.* **14** 1060–5, 1066–70, 1071–4

Wood D W and Griffiths H P 1973 *J. Math. Phys.* **14** 1715–22